A Statistical Model for Turing Patterns in Reaction-Diffusion Systems

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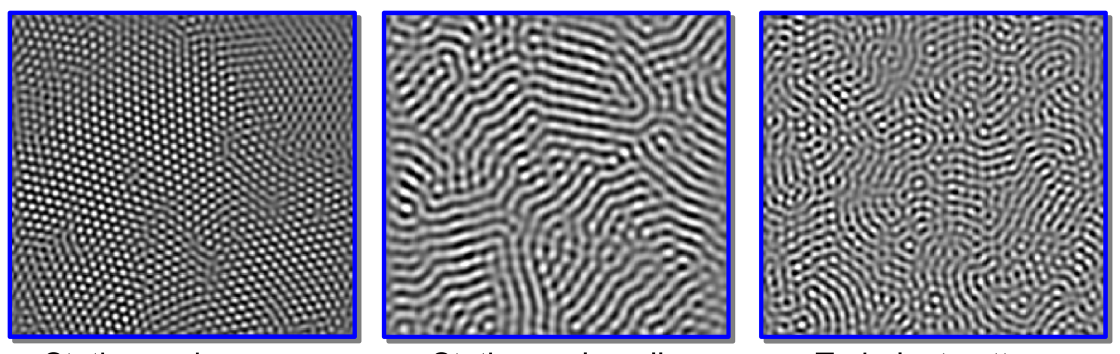
Introduction

Experimental CIMA Patterns

Quasi 2D patterns in continuously fed chlorite-iodide-malonic acid (CIMA) reaction


Stationary hexagons | Stationary lamellae | Turbulent pattern

- Non-equilibrium quasi-2D reaction-diffusion process
- Pattern selection depends on iodide, malonic acid concentration and pH value
- Grayscale proportional to iodide concentration u

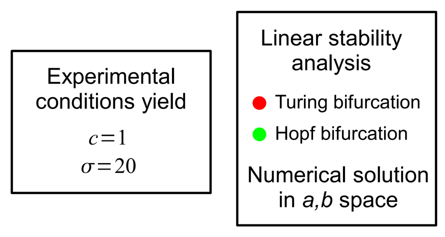
Q. Ouyang, H. L. Swinney, Chaos **1**, 411, 1991

Lengyel-Epstein (LE) Model

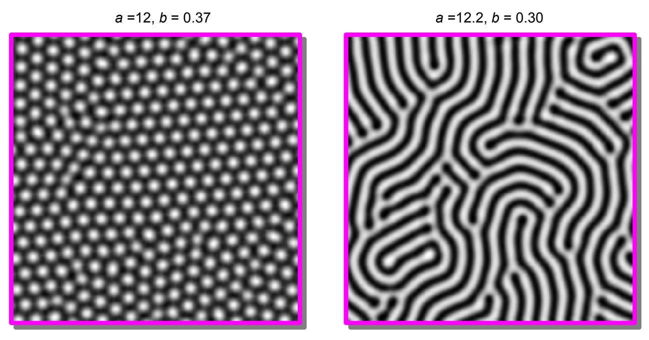
Two component reaction-diffusion equation

$$\frac{\partial}{\partial t}u(x,y,t)=\nabla^2 u+a-u-\frac{4uv}{1+u^2},\quad \frac{\partial}{\partial t}v(x,y,t)=\sigma\left[c\nabla^2 v+b\left(u-\frac{uv}{1+u^2}\right)\right]\qquad u\equiv[I^-],\ v\equiv[ClO_2^-]$$

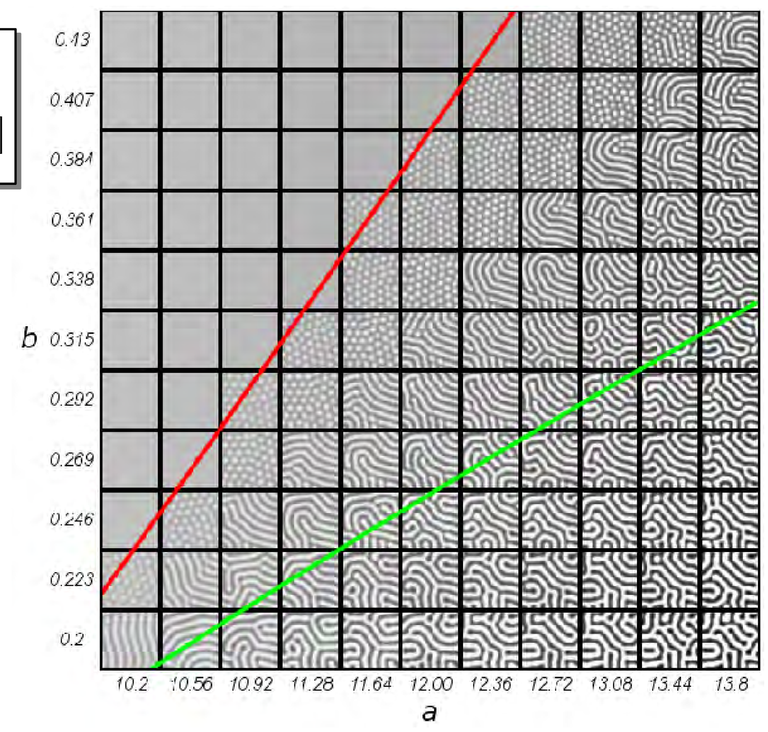
- c depends on diffusivity of u,v
- σ depends on starch concentration
- a and b depend on supply concentration of iodide and malonic acid



Qualitative agreement to stationary CIMA patterns



u, hexagons | u, lamellae

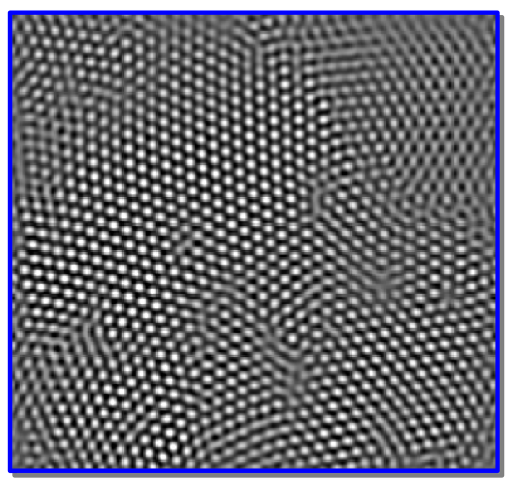


I. Lengyel, I. R. Epstein, PNAS **89**, 3977, 1992

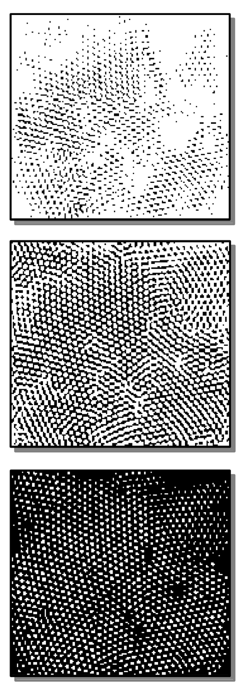
Morphological Analysis

Morphological Analysis using Minkowski Functionals

Quantitative characterisation of concentration profiles



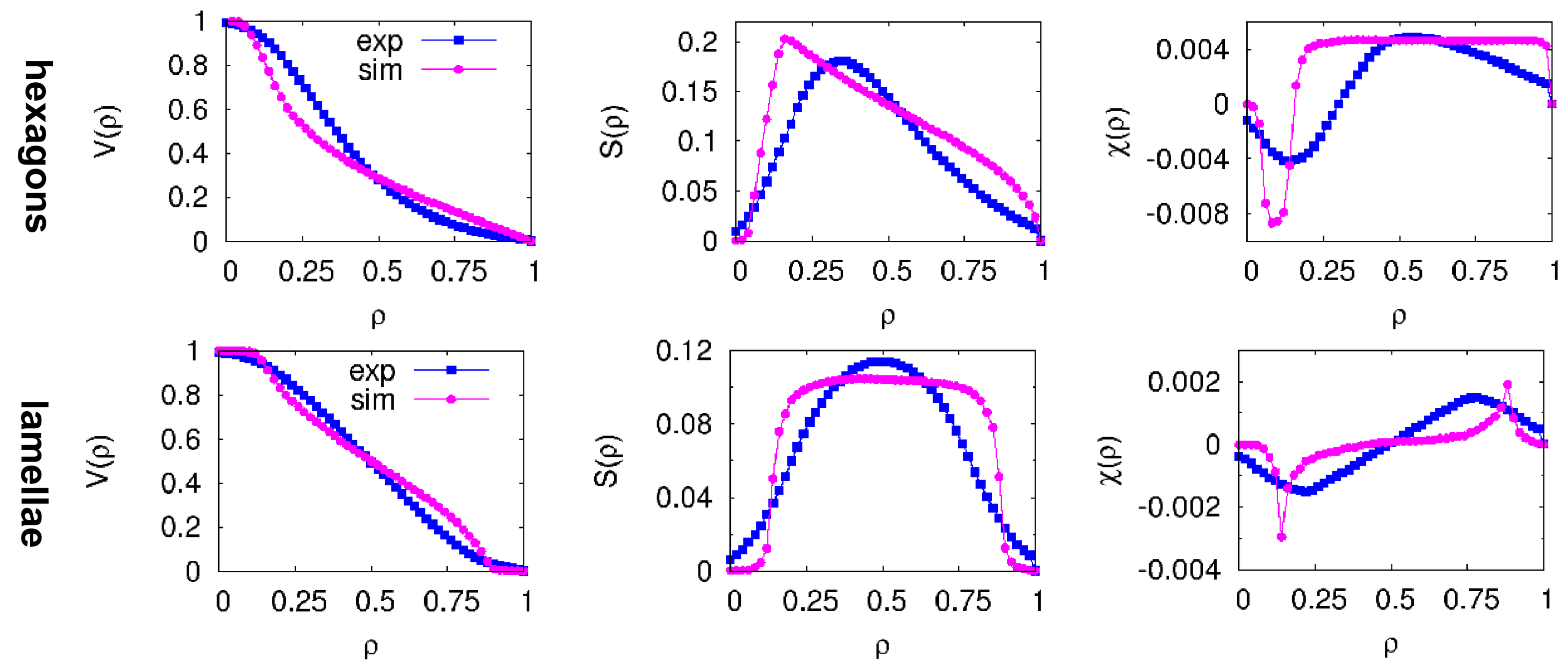
Threshold ρ



Morphological analysis using integral geometric measures

Minkowski Functionals

Area	$V(\rho)$
Perimeter	$S(\rho)$
Euler-Characteristic	$\chi(\rho)$

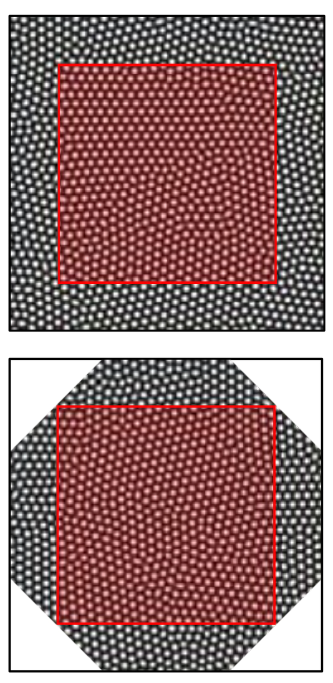


Significant differences between experiment and mathematical model

K. Mecke, Phys. Rev. E **53**, 4794,1996

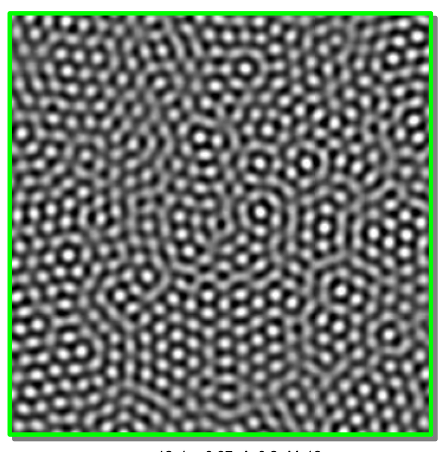
Linear Superposition

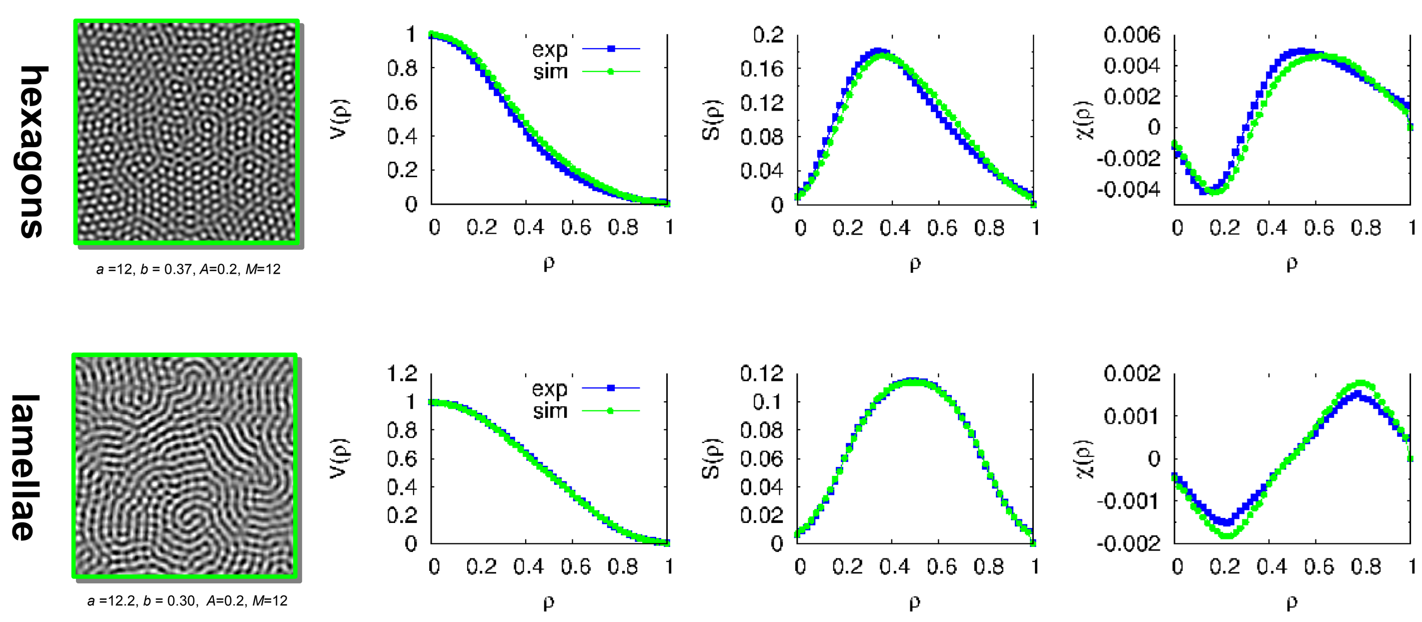
Linear Superposition of Simulated Patterns and Morphological Results



Linear superposition of M equivalent simulated and randomly rotated patterns

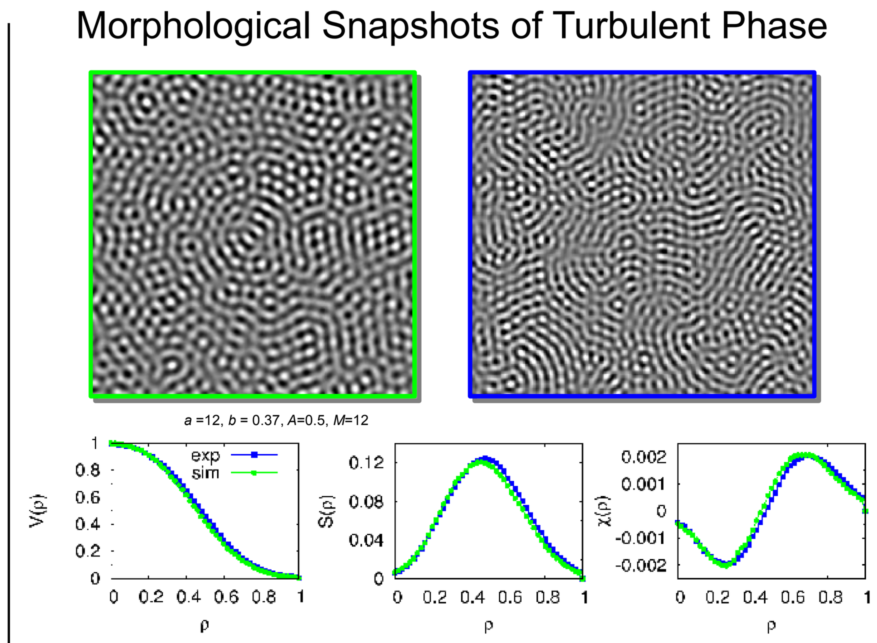
$$u_{sp}=u_0+A\cdot\sum_{i=1}^{M}u_i,\quad A\in[0,1]$$


a =12, b = 0.37, A=0.2, M=12



Good morphological agreement

Morphological Snapshots of Turbulent Phase



consistent morphology → Model for turbulent phase

Interacting Pattern Gas

Statistical Theory

Energy Functional

$$F[u]=\int_V dr\left(\frac{1}{2}\sum_{i,j}D_{ij}\nabla u_i\nabla u_j+f(u)\right)$$

For a superposition of basic patterns $u=\frac{1}{N}\sum_{\nu=1}^{N}u^\nu$

$$\Delta F=F[u]-\frac{1}{N}\sum_{\nu=1}^{N}F[u^\nu]$$

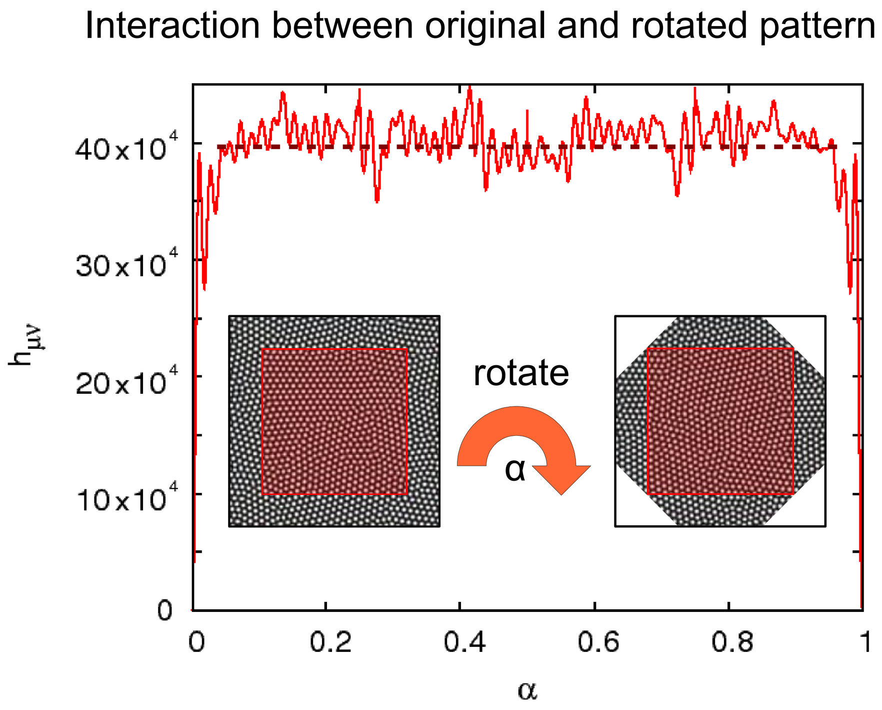
In 2nd order approximation

$$\Delta F=\frac{1}{2N^2}\sum_{(\nu\mu)}h_{\nu\mu}$$

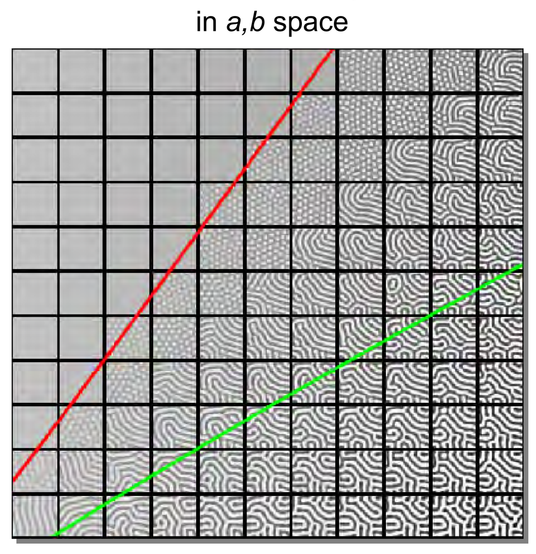
where $h_{\nu\mu}$ = interaction energy for a pair of patterns ν,μ

Interaction Energy and Phase Space Dependence

Interaction between original and rotated pattern

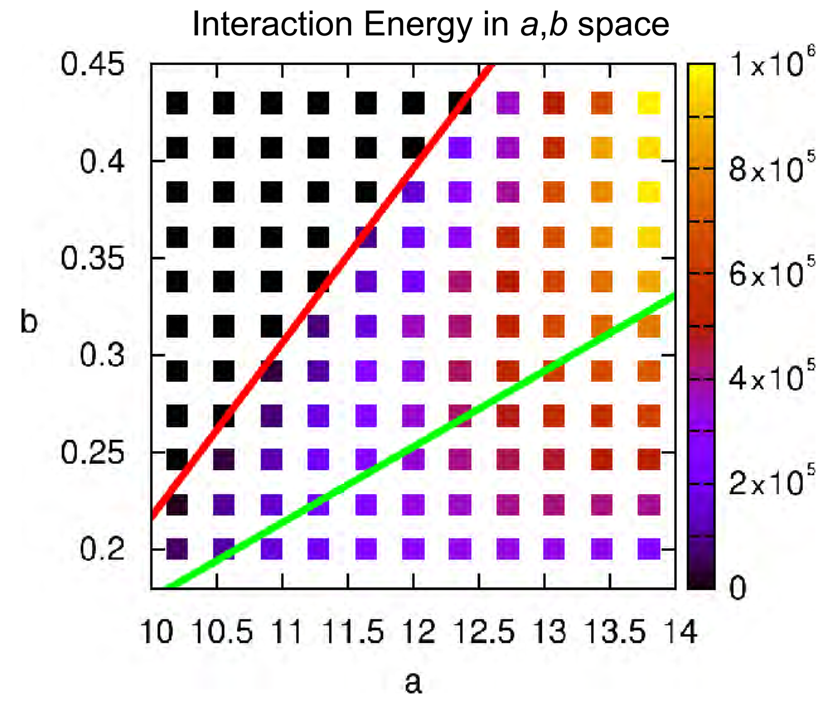


Plateau value in a,b space



Superposition possible for small ΔF

Interaction Energy in a,b space



Outlook

Summary

- Deterministic reaction-diffusion models only partially capture pattern formation in the CIMA reaction
- Simple model based on superposition of basic LE patterns with good morphological agreement to the experiment
- Morphological interpretation of turbulent patterns as statistical superposition of basic patterns

Outlook

- Detailed analysis of the interacting pattern gas model
- Experimental verification of parameter space dependence

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